# ATTITUDE STABILITY OF SPINNING SATELLITES

by

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January 1980

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# INTRODUCTION

Since the attitude instability experience by Explorer 1, many papers have been written on the effects of internal dissipation on the attitude stability of spinning satellites. In the engineering literature, stability analysis is restricted to the variational or linearized perturbational equations, despite the fact that spinning satellites are almost always critical cases in the Liapunov-Poincaré stability theory. This is certainly true in the case of dual spin satellites, which have the further complication that the linearized perturbational equations have periodic coefficients.

The purpose of this note is to treat some problems of attitude stability of spinning satellites in a rigorous manner and to show that, with certain restrictions, the linearized stability analysis correctly predicts the attitude stability of spinning satellites.

## 1. Detumbling of a Spacecraft Using Passive Torsional Dampers

### Formulation of Problem

Consider a spacecraft which is designed to spin about axis 1, the axis of maximum moment of inertia, to provide an artificial gravity field for the crew. Attached to the spacecraft on axes 2 and 3 are torsional dampers, consisting of inertia wheels of polar moment of inertia $J_i$, $(i=2,3)$ with torsional springs with restoring torque $K_i f(\theta_i)$ and damping torque $D_i \dot{\theta}_i$. Let $I_1$, $I_2$, $I_3$ be the moments of inertia of the spacecraft about the 1,2 and 3 axes respectively, including the moments of inertia of the dampers

$$\text{Let} \quad I_1 > I_2 \geq I_3 >> J_i \qquad i=2,3$$

Suppose that owing to collision with another spacecraft, which is attempting to dock with the first spacecraft, a tumbling motion results. Let $\omega_1$, $\omega_2$, $\omega_3$ be the angular velocities of the tumbling motion with respect to the body-fixed axes 1,2 and 3 respectively. For the safety and comfort of the crew, and to make docking possible, the spacecraft must be detumbled and returned to a state of simple spin about the 1 axis.

### Equations of Motion

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) + J_3\omega_2\dot{\theta}_3 - J_2\omega_3\dot{\theta}_2 = 0 \tag{1.1}$$

$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) + J_2\ddot{\theta}_2 - J_3\omega_1\dot{\theta}_3 = 0 \tag{1.2}$$

$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) + J_3\ddot{\theta}_3 + J_2\omega_1\dot{\theta}_2 = 0 \tag{1.3}$$

$$J_2(\ddot{\theta}_2 + \dot{\omega}_2) + D_2\dot{\theta}_2 + K_2 f(\theta_2) = 0 \tag{1.4}$$

$$J_3(\ddot{\theta}_3 + \dot{\omega}_3) + D_3\dot{\theta}_3 + K_3 f(\theta_3) = 0 \tag{1.5}$$

Where:

$$f(\theta)=-f(-\theta)$$

$$\theta f(\theta)>0 \qquad \theta\neq 0$$

$$\lim_{\theta\to 0}\frac{f(\theta)}{\theta}=1\ , \quad \int_0^\theta f(y)dy=F(\theta)>0 \qquad \theta\neq 0 \tag{1.6}$$

$$D_i,K_i>0 \qquad i=2,3$$

Global Stability

Let

$$V=\frac{1}{2}[I_1\omega_1^2+(I_2-J_2)\omega_2^2+(I_3-J_3)\omega_3^2+J_2(\omega_2+\dot\theta_2)^2+J_3(\omega_3+\dot\theta_3)^2]+K_2F(\theta_2)+K_3F(\theta_3) \tag{1.7}$$

the function $V$ is clearly positive definite

$$\dot V=I_1\omega_1\dot\omega_1+I_2\omega_2\dot\omega_2+I_3\omega_3\dot\omega_3+J_2(\dot\omega_2\dot\theta_2\ +\omega_2\ddot\theta_2+\dot\theta_2\ddot\theta_2)+K_2f(\theta_2)\dot\theta_2$$

$$+J_3(\dot\omega_3\dot\theta_3\ +\omega_3\ddot\theta_3+\dot\theta_3\ddot\theta_3)+K_3f(\theta_3)\dot\theta_3 \tag{1.8}$$

Using equations (1.1),(1.2),(1.3),(1.4) and (1.5) to evaluate $\dot V$ along the trajectories of the motion, we have:

$$\dot V=-[D_2\dot\theta_2^2+D_3\dot\theta_3^2]\leq 0 \tag{1.9}$$

The function $V$ is positive definite and its time derivative along the trajectories of the motion is negative semi-definite, therefore $V$ is a Liapunov function and the tumbling motion is globally Liapunov stable. We note that $\dot V$ is only semi-definite and vanishes when $\dot\theta_2=\dot\theta_3=0$. Equations (1.4),(1.5) show that $\ddot\theta_2$ and $\ddot\theta_3$ are not zero unless,

(a) $\dot{\omega}_2=\dot{\omega}_3=0$ and $\theta_2=\theta_3=0$

or (b) $K_2f(\theta_2)=-J_2\dot{\omega}_2$ and $K_3f(\theta_3)=-J_3\dot{\omega}_3$

Examination of equations (1.1),(1.2) and (1.3) shows that condition (b) cannot be satisfied in general unless $\dot{\omega}_1=\dot{\omega}_2=\dot{\omega}_3=0$ , $\ddot{\theta}_i=\dot{\theta}_i=\theta_i=0$ , $i=2,3$ and one of the following conditions hold

i) $\omega_1\neq 0$ , $\omega_2=\omega_3=0$

ii) $\omega_2\neq 0$ $\omega_1=\omega_3=0$

iii) $\omega_3\neq 0$ , $\omega_1=\omega_2=0$

This set of conditions are simply the equilibrium solutions of the set of equations (1.1),(1.2),(1.3),(1.4) and (1.5). With this exception, $\dot{\theta}_2=\dot{\theta}_3=0$, only on a set of measure zero. Thus, using (1.9),

$$V(t)-V(0)=-\int_0^t[D_2\dot{\theta}_2^2+D_3\dot{\theta}_3^2]dt<0 \ . \tag{1.10}$$

Hence, the function $V(t)$ decreases along the trajectories of the motion. $V(t)$ must therefore tend to a limit corresponding to one of the equilibrium solutions. The particular limit to which all motions ultimately tend for large time is determined by the stability of the equilibrium solutions. Clearly all motions will tend in the limit to the largest invariant set, which corresponds to a stable equilibrium solution.

## Stability of the Equilibrium Solutions

Examination of equations (1.1) through (1.5) shows that there are three equilibrium solutions.

$\dot{\omega}_i=0$ , $i=1,2,3$ ; $\ddot{\theta}_j=\dot{\theta}_j=\theta_j=0$ , $j=2,3$

i) $\omega_1\neq 0$ , $\omega_2=\omega_3=0$

ii) $\omega_2\neq 0$ , $\omega_1=\omega_3=0$

iii) $\omega_3\neq 0$ , $\omega_1=\omega_2=0$

Case (i)

$\omega_1=\omega_{10}$ , $\omega_2=\omega_3=0$

Let

$\omega_1=\omega_{10}+\xi$ $\omega_2=\eta$ , $\omega_3=\zeta$

$\theta_j=\alpha_j$ $j=2,3$

Perturbing about the steady state solution and retaining only the linear terms in the equations of motion.

$$I_1\dot{\xi}=0 \tag{1.11}$$

$$I_2\dot{\eta}+\omega_{10}\zeta(I_1-I_3)+J_2\ddot{\alpha}_2-J_3\omega_{10}\dot{\alpha}_2=0 \tag{1.12}$$

$$I_3\dot{\zeta}+\omega_{10}\eta(I_2-I_1)+J_3\ddot{\alpha}_3+J_2\omega_{10}\dot{\alpha}_2=0 \tag{1.13}$$

$$J_2(\ddot{\alpha}_2+\dot{\eta})+D_2\dot{\alpha}_2+K_2\alpha_2 \qquad =0 \tag{1.14}$$

$$J_3(\ddot{\alpha}_3+\dot{\zeta})+D_3\dot{\alpha}_3+K_3\alpha_3 \qquad =0 \tag{1.15}$$

Let

$$\frac{D_2}{J_2}=\beta_2 \quad ; \quad \frac{K_2}{J_2}=p_2^2$$

$$\frac{D_3}{J_3}=\beta_3 \quad ; \quad \frac{K_3}{J_3}=p_3^2$$

Define

$$\Omega_1^2 = \frac{\omega_{10}^2 (I_1 - I_2)(I_1 - I_3)}{I_2 I_3}$$

The characteristic equation for the system of linear differential equations (1.11) through (1.15) is

$$\lambda \left\{ \left[ \left(1 - \frac{J_2}{I_2}\right)\lambda^3 + \beta_2\lambda^2 + p_2^2\lambda \right] \left[ \left(1 - \frac{J_3}{I_3}\right)\lambda^3 + \beta_3\lambda^2 + p_3^2\lambda \right] \right.$$

$$\left. + \Omega_1^2 \left[ \left(1 + \frac{J_2}{I_1 - I_2}\right)\lambda^2 + \beta_2\lambda + p_2^2 \right] \left[ \left(1 + \frac{J_3}{I_1 - I_3}\right)\lambda^2 + \beta_3\lambda + p_3^2 \right] \right\} = 0 \tag{1.16}$$

Let

$$\beta_i = \left(1 - \frac{J_i}{I_i}\right)\beta$$

$$p_i^2 = \left(1 - \frac{J_i}{I_i}\right)p^2 \tag{1.17}$$

$$i = 2, 3$$

If $J_2$ and $J_3$ are selected such that

$$\left.\begin{aligned} &\left( \frac{1 + \frac{J_i}{I_1 - I_2}}{1 - \frac{J_i}{I_i}} \right) = (1 + \mu) \\ &i = 2, 3 \qquad \mu > 0 \end{aligned}\right\} \tag{1.18}$$

The characteristic equation (1.16) becomes

$$\lambda\{[\lambda^3+\beta\lambda^2+p^2\lambda]^2+\Omega^2[(1+\mu)\lambda^2+\beta\lambda+p^2]^2\}=0 \tag{1.19}$$

which may be written in the form

$$f(\lambda)=\lambda g_1(\lambda)g_2(\lambda)=0 \tag{1.20}$$

where

$$\begin{aligned} g_1(\lambda)&=i[\lambda^3+\beta\lambda^2+p^2\lambda]+\Omega[(1+\mu)\lambda^2+\beta\lambda+p^2] \\ g_2(\lambda)&=-i[\lambda^3+\beta\lambda^2+p^2\lambda]+\Omega[(1+\mu)\lambda^2+\beta\lambda+p^2] \end{aligned} \tag{1.21}$$

Using Cauchy's Principle of the argument, or Nyquist's criteron, it is easily shown that $g_i(\lambda)$, $i=1,2$, have zeros only in the left half $\lambda$ plane. Thus

$$\lambda_1=0\ ,\ \mathrm{Re}\lambda_i<0 \qquad i\in(2,7)\ .$$

This is clearly one of the critical cases in Liapunov stability theory, however, using Theorem AIII of the appendix, we see that the full perturbation equations are Liapunov stable. Thus, the equilibrium solution (i) is stable.

Case (ii)

$$\omega_2=\omega_{20}\neq 0\ ,\ \omega_1=\omega_3=0$$

Let

$$\omega_1=\xi\ ,\ \omega_2=\omega_{20}+\eta\ ,\ \omega_3=\zeta$$

$$\theta_i=\alpha_i \qquad i=2,3$$

Perturbing about the equilibrium solution and retaining only the linear terms in the equations of motion, we have:

$$I_1\dot{\xi}+\omega_{20}\zeta(I_3-I_2)+J_3\omega_{20}\dot{\alpha}_3=0 \tag{1.22}$$

$$I_2\dot{\eta}+J_2\ddot{\alpha}_2 \qquad =0 \tag{1.23}$$

$$I_3\dot{\zeta}+\omega_{20}\xi(I_2-I_1)+J_3\ddot{\alpha}_3 \qquad =0 \tag{1.24}$$

$$J_2(\ddot{\alpha}_2+\dot{\eta})+D_2\dot{\alpha}_2+K_2\alpha_2 \qquad =0 \tag{1.25}$$

$$J_3(\ddot{\alpha}_3+\dot{\zeta})+D_3\dot{\alpha}_3+K_3\alpha_3 \qquad =0 \tag{1.26}$$

Let

$$\left.\begin{aligned} &\Omega_2^2=\frac{\omega_{20}^2(I_2-I_3)(I_1-I_2)}{I_1I_3} \\ &\beta_i=\frac{D_i}{J_i}\ ;\ p_i^2=\frac{K_i}{J_i} \qquad i=2,3 \end{aligned}\right\} \tag{1.27}$$

The characteristic equation for this set of linear differential equations is:

$$\lambda\left[\lambda^2\left(1-\frac{J_2}{I_2}\right)+\beta_2\lambda+p_2^2\right]\left[\lambda^4\left(1-\frac{J_3}{I_3}\right)+\beta_3\lambda^3-\beta_3\Omega_2^2\lambda-\Omega_2^2p_3^2\right.$$
$$\left.+\left(p_3^2-\Omega_2^2\left(1+\frac{J_3}{I_2-I_3}\right)\right)\lambda^2\right]=0 \tag{1.28}$$

Application of Cauchy's principle of the argument or Nyquist's stability criterion immediately shows that

$$\left.\begin{aligned} &\lambda_1=0 \quad \lambda_2>0 \quad \lambda_3<0 \\ &\mathrm{Re}\lambda_i<0 \qquad i=4,5,6,7 \end{aligned}\right\} \tag{1.29}$$

Since $\lambda_2>0$ , application of Theorem II shows that the full perturbational equations are unstable in the sense of Liapunov stability theory.

Case (iii)

$$\omega_3=\omega_{30}\neq 0 \qquad \omega_1=\omega_2=0$$

Let

$$\omega_1=\xi \ , \ \omega_2=\eta \ , \ \omega_3=\omega_{30}+\zeta$$

Perturbing about the equilibrium solution and retaining only the linear terms in the equations of motion, we have:

$$I_1\dot{\xi}+\omega_{30}\eta(I_3-I_2)-J_2\omega_{30}\dot{\alpha}_2 =0 \tag{1.30}$$

$$I_2\dot{\eta}+\omega_{30}\xi(I_1-I_3)+J_2\ddot{\alpha}_2 =0 \tag{1.31}$$

$$I_3\dot{\zeta}+J_3\ddot{\alpha}_3 =0 \tag{1.32}$$

$$J_2(\ddot{\alpha}_2+\dot{\eta})+D_2\dot{\alpha}_2+K_2\alpha_2 =0 \tag{1.33}$$

$$J_3(\ddot{\alpha}_3+\dot{\zeta})+D_3\dot{\alpha}_3+K_3\alpha_3 =0 \tag{1.34}$$

Let

$$\left.\begin{aligned} &\Omega_3^2=\frac{\omega_{30}^2(I_1-I_3)(I_2-I_3)}{I_1I_2} \\ &\beta_i=\frac{D_i}{J_i} \ ; \ p_i^2=\frac{K_i}{J_i} \ , \quad i=2,3 \end{aligned}\right\} \tag{1.35}$$

The characteristic equation for this set of linear differential equations is:

$$\lambda\left[\lambda^2\left(1-\frac{J_3}{I_3}\right)+\beta_3\lambda+p_3^2\right]\left[\lambda^4\left(1-\frac{J_2}{I_3}\right)+\beta_2\lambda^3+\Omega_3^2\beta_2\lambda\right.$$

$$\left.+\left(p_2^2+\Omega_2^2\left(1-\frac{J_2}{I_2-I_3}\right)\right)\lambda^2+\Omega_2^2p_2^2\right]=0 \qquad (1.36)$$

Application of Caughy's principle of the argument, or Nyquist's stability criterion immediately shows that (1.36) has roots:

$$\left.\begin{array}{l}\lambda_1=0\ ,\ \mathrm{Re}\lambda_i>0 \qquad i=2,3 \\ \\ \mathrm{Re}\lambda_i<0 \qquad i=4,5,6,7\end{array}\right\} \qquad (1.37)$$

Since there are two eigenvalues whose real parts are positive, application of Theorem AII shows that the full perturbational equations are unstable in the sense of Liapunov stability theory. Thus we see that the only stable equilibrium solution is that corresponding to Case (i) $\omega_1\neq 0$ , $\omega_2=\omega_3=0$. From the analysis of global stability we know that the function $V(t)$ (1.7) decreases along the trajectories of the motion and tends to a limit corresponding to a stable equilibrium solution, the only stable equilibrium solution is that corresponding to spin about the 1 axis, the axis of maximum moment of inertia. Thus we have shown rigorously that it is possible to detumble a spacecraft using only passive torsional dampers. Edwards and Kaplan (1) have treated the problem of automatic detumbling of a spacecraft using the motion of a servo-controlled internal mass. Their treatment is heuristic rather than rigorous.

## 2. Stability of a Dual Spin Satellite

The stability of dual spin satellites has been examined by a number of authors, however, in the case where the rotor and the platform both exhibit internal dissipation, the analytical solution was first presented by Sarychev and Sazonov (2) who used Floquet Theory. In this note the effects of internal dissipation will be modelled by torsional dampers in both rotor and platform. It will be shown that the linearized stability analysis is rigorously justified and it will also be shown that the linearized stability analysis can be performed quite simply by using Lagranges method of variation of parameter.

## Formulation of the Problem

The dual spin satellite consists of two rigid bodies with a common axis of rotation (axis 3)

Let the axial moment of inertia of the rotor be $J$.

Let the total axial moment of inertia of the satellite be $C$ (rotor plus platform, plus dampers)

Let the total equatorial moment of inertia of the satellite be $A$ (including the dampers)

Let $I_b$ and $I_b'$ be the polar moments of inertia of the dampers wheels on the platform and rotor respectively

Let $\overline{K}_1$ and $\overline{K}_2$ be the damping and stiffness parameters of the torsional damper on the platform. Let $\overline{K}_1'$ and $\overline{K}_2'$ be the corresponding parameters for the rotor damper:

Let $\omega_1$ and $\omega_2$ be the angular velocities of the platform with respect to the 1 and 2 axes respectively. Let $\omega_3$ be the angular velocity of the platform about the 3 axes

Let $\dot{\psi}$ be the angular velocity of the rotor about the $3'$ axis relative to the platform, where the angle $\psi$ (measured about the $3'$ axis) defines the orientation of the body fixed axes of the rotor with respect to body fixed axis of the platform.

Let $T_B$ be the frictional torque of the rotor bearings

Let $T_M$ be the torque of the despin motor.

Equations of Motion

$$\left.\begin{aligned}
&J(\dot{\omega}_3+\ddot{\psi})-I_b'\dot{\theta}[\omega_2\cos\psi-\omega_1\sin\psi]+T_B-T_M=0\\
&C\dot{\omega}_3+J\ddot{\psi}-I_b'\dot{\theta}[\omega_2\cos\psi-\omega_1\sin\psi]-I_b\omega_2\dot{\phi}=0\\
&A\dot{\omega}_1+[(C-A)\omega_3+J\dot{\psi}]\omega_2+I_b'[\ddot{\theta}\cos\psi-\dot{\theta}(\omega_3+\dot{\psi})\sin\psi]+I_b\ddot{\phi}=0\\
&A\dot{\omega}_2-[(C-A)\omega_3+J\dot{\psi}]\omega_1+I_b'[\ddot{\theta}\sin\psi+\dot{\theta}(\omega_3+\dot{\psi})\cos\psi]+I_b\omega_3\dot{\phi}=0\\
&I_b'\ddot{\theta}+\overline{K}_1'\dot{\theta}+\overline{K}_2'\theta+I_b'[(\dot{\omega}_1+\omega_2\dot{\psi})\cos\psi+(\dot{\omega}_2-\omega_1\dot{\psi})\sin\psi]=0\\
&I_b\ddot{\phi}+\overline{K}_1\dot{\phi}+\overline{K}_2\phi+I_b\dot{\omega}_1=0
\end{aligned}\right\}\quad(2.1)$$

Where $\phi$, $\theta$ are the rotation angles of the torsional dampers on the platform and rotor respectively.

Steady State Solutions

If the torque of the despin motor just balances the bearing friction torque when $\dot{\psi}=\sigma$ , then the steady state solution is:

$$\left.\begin{aligned}
&\omega_3=\Omega\ ,\ \dot{\psi}=\sigma\ ,\ \psi=\sigma t\\
&\theta=\phi=\dot{\theta}=\dot{\phi}=\omega_1=\omega_2=0
\end{aligned}\right\}\quad(2.2)$$

Linearized Stability Equations

Let

$$\left.\begin{aligned}
&\omega_3=\Omega+\sigma\xi \;;\; \tau=\sigma t \;;\; r=\frac{\Omega}{\sigma}\\
&\psi_1=\psi=\sigma(t+\eta) \;;\; \psi_2=\dot{\psi}_1=\sigma(1+\zeta)\\
&\frac{\omega_1}{\sigma}=v_1 \;;\; \frac{\omega_2}{\sigma}=v_2 \;;\; T_B-T_M=\beta\sigma\zeta\\
&\mu=\frac{I_b}{A}\;,\; \mu'=\frac{I_b'}{A}\;,\; K_1=\frac{\overline{K}_1}{I_b\sigma} \;;\; K_1'=\frac{\overline{K}_1'}{I_b'\sigma}\\
&K_2=\frac{\overline{K}_2}{I_b\sigma^2} \;;\; K_2'=\frac{\overline{K}_2}{I_b'\sigma^2}
\end{aligned}\right\} \tag{2.3}$$

In Appendix C, it is shown that the steady state solution is stable for sufficiently small perturbations, provided the following conditions are satisfied

1) $\beta>0$ (2.4)

and the systems of equations:

$$\left.\begin{aligned}
&\frac{dv_1}{dc} + \lambda v_2 + \mu \frac{d^2\psi}{d\tau^2} + \mu'\left[\frac{d^2\theta}{d\tau^2}\cos\tau - (1+r)\frac{d\theta}{d\tau}\sin\tau\right] = 0\\
&\frac{dv_2}{d\tau} - \lambda v_2 + \mu r \frac{d\psi}{d\tau} + \mu'\left[\frac{d^2\theta}{d\tau^2}\sin\tau + (1+r)\frac{d\theta}{d\tau}\cos\tau\right] = 0\\
&\frac{d^2\theta}{d\tau^2} + K_1' \frac{d\theta}{d\tau} + K_2'\theta + \left[\left(\frac{dv_1}{d\tau} + v_2\right)\cos\tau + \left(\frac{dv_2}{d\tau} - v_1\right)\sin\tau\right] = 0\\
&\frac{d^2\phi}{d\tau^2} + K_1 \frac{d\phi}{d\tau} + K_2\phi + \frac{dv_1}{d\tau} = 0
\end{aligned}\right\} \quad (2.5)$$

is Liapunov asymptotically stable.

Equations (2.5) have periodic coefficients and may be rewritten in standard form as

$$\frac{d\underline{p}}{d\tau} = A_3(\tau)\underline{p} \qquad (2.6)$$

where

$$A_3(\tau + \pi) = A_3(\tau) \qquad (2.7)$$

and

$$\underline{p} = \begin{pmatrix} v_1 \\ v_2 \\ \phi \\ \frac{d\phi}{d\tau} \\ \theta \\ \frac{d\theta}{d\tau} \end{pmatrix} \qquad (2.8)$$

The stability of equation (2.6) may then be investigated by using Floquet theory, as was done by Sarychev and Sazonov (1). Alternatively the stability of equations (2.5) may be investigated directly in the case where $\mu,\mu'$ are small, by using Lagrange's method of variation of parameters.

If $\varepsilon = \mathrm{Max}(\mu,\mu')$ and $\varepsilon \ll 1$, then equations (2.5) are of the type treated in Appendix B.

Let

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} \cos\lambda\tau & \sin\lambda\tau \\ \sin\lambda\tau & -\cos\lambda\tau \end{bmatrix} \underline{a}(\tau) \tag{2.9}$$

where

$$\underline{a}(t) = \begin{pmatrix} a_1(\tau) \\ a_2(\tau) \end{pmatrix}$$

Substituting into equations (2.5) we have:

$$\left.\begin{aligned} \frac{da_1}{d\tau}\cos\lambda\tau + \frac{da_2}{d\tau}\sin\lambda\tau &= -\mu'\,\frac{d^2\theta}{d\tau^2}\cos\tau + \mu'(1+r)\frac{d\theta}{d\tau}\sin\tau - \mu\,\frac{d^2\theta}{d\tau^2} \\ \frac{da_1}{d\tau}\sin\lambda\tau - \frac{da_2}{d\tau}\cos\lambda\tau &= -\mu'\,\frac{d^2\theta}{d\tau^2}\sin\tau - \mu'(1+r)\frac{d\theta}{d\tau}\cos\tau - \mu r\,\frac{d\phi}{d\tau} \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned}
&\frac{d^2\theta}{d\tau^2} + K_1' \frac{d\theta}{d\tau} + K_2'\theta = (\lambda-1)\left[a_1\cos(\lambda-1)\tau - a_2\sin(\lambda-1)\tau\right] \\
&\qquad\qquad - \left[\frac{da_1}{d\tau}\cos(\lambda+1)\tau + \frac{da_2}{d\tau}\sin(\lambda+1)\tau\right] \\
&\frac{d^2\phi}{d\tau^2} + K_1 \frac{d\phi}{d\tau} + K_2\phi = \lambda\left[a_1\sin\lambda\tau - a_2\cos\lambda\tau\right] \\
&\qquad\qquad - \left[\frac{da_1}{d\tau}\cos\lambda\tau + \frac{da_2}{d\tau}\sin\lambda\tau\right]
\end{aligned}\right\} \tag{2.11}$$

Using equations (2.10) to solve for $da_1/d\tau$ and $da_2/d\tau$

$$\left.\begin{aligned}
\frac{da_1}{d\tau} &= -\mu' \frac{d^2\theta}{d\tau^2}\cos(\lambda-1)\tau - (1+r)\mu' \frac{d\theta}{d\tau}\sin(\lambda-1)\tau \\
&\qquad - \mu \frac{d^2\phi}{d\tau^2}\cos\lambda\tau - \mu r \frac{d\phi}{d\tau}\sin\lambda\tau \\
\frac{da_2}{d\tau} &= -\mu' \frac{d^2\theta}{d\tau^2}\sin(\lambda-1)\tau + (1+r)\mu' \frac{d\theta}{d\tau}\cos(\lambda-1)\tau \\
&\qquad - \frac{d^2\phi}{d\tau^2}\sin\lambda\tau + \mu r \frac{d\phi}{d\tau}\cos\lambda\tau
\end{aligned}\right\} \tag{2.12}$$

Clearly, if $d\theta/d\tau$ , $d^2\theta/d\tau$ , $d\theta/d\tau$ , $d^2\phi/d\tau^2$ are bounded, since $\varepsilon \ll 1$ $\varepsilon = \mathrm{Max}(\mu,\mu')$, hence

$$\frac{\left|\frac{da_1}{d\tau}\right| + \left|\frac{da_2}{d\tau}\right|}{|a_1| + |a_2|} \sim O(\varepsilon) \tag{2.13}$$

Thus $a_1(\tau)$ and $a_2(\tau)$ are slowly varying functions of $\tau$, hence in equations (2.11) we may neglect the terms $da_1/d\tau$ , $da_2/d\tau$ in comparison to $a_1$ and $a_2$. We may further treat $a_1(\tau)$ and $a_2(\tau)$ as "constant", provided $K_1'$ and $K_1$ are not too small.

Thus the "steady State" solutions of equations (2.11) are:

$$\left.\begin{aligned}\theta(\tau)&=C(\tau)\cos(\lambda-1)\tau+D(\tau)\sin(\lambda-1)\tau+O(\varepsilon)\\ \phi(\tau)&=E(\tau)\cos\lambda\tau+F(\tau)\sin(\lambda-1)\tau+O(\varepsilon)\end{aligned}\right\}\quad(2.14)$$

where

$$\left.\begin{aligned}C(\tau)&=\frac{-(\lambda-1)[K_2'-(\lambda-1)^2)a_2(\tau)+(\lambda-1)K_1'a_1(\tau)]}{[K_2'-(\lambda-1)^2]^2+[K_1'(\lambda-1)]^2}\\ D(\tau)&=\frac{-(\lambda-1)[(\lambda-1)K_1'a_2(\tau)-(K_2-(\lambda-1)^2)a_1(\tau)]}{[K_2'-(\lambda-1)^2]^2+[K_1'(\lambda-1)]^2}\\ E(\tau)&=\frac{-\lambda[(K_2-\lambda)^2a_2(\tau)+\lambda K_1a_1(\tau)]}{[K_2-\lambda^2]^2+[K_1\lambda]^2}\\ F(\tau)&=\frac{-\lambda[\lambda K_1a_2(\tau)-(K_2-\lambda^2)a_1(\tau)]}{[K_2-\lambda^2]^2+[K_1\lambda]^2}\end{aligned}\right\}\quad(2.15)$$

Substituting equations (2.14) into equations (2.12), treating $a_1(\tau)$ and $a_2(\tau)$ as constants. Consistent with this, we retain the time averaged coefficients of $C(\tau)$ , $D(\tau)$ , $E(\tau)$ and $F(\tau)$ in the resulting equations, thus we have

$$\left.\begin{aligned}\frac{da_1}{d\tau} &= \frac{\mu'}{2}[(\lambda-1)(\lambda+r)]C+\frac{\mu}{2}[\lambda(\lambda+r)]E+0(\varepsilon^2)\\ \frac{da_2}{d\tau} &= \frac{\mu'}{2}[(\lambda-1)(\lambda+r)]D+\frac{\mu}{2}[\lambda(\lambda+r)]F+0(\varepsilon^2)\end{aligned}\right\} \tag{2.16}$$

Using equation (2.15) to substitute for C, D, E and F in terms of $a_1(\tau)$ and $a_2(\tau)$.

$$\left.\begin{aligned}\frac{da_1}{d\tau} &=\alpha a_1-\beta a_2+0(\varepsilon^2)\\ \frac{da_2}{d\tau} &=\alpha a_2+\beta a_1+0(\varepsilon^2)\end{aligned}\right\} \tag{2.17}$$

where

$$\alpha=-[\lambda+r]\left[\frac{\frac{\mu'}{2}(\lambda-1)^3K_1'}{[K_1'-(\lambda-1)^2]^2+[K_1'(\lambda-1)]^2}+\frac{\frac{\mu}{2}\lambda^3K_1}{[K_2-\lambda^2]^2+[K_1\lambda]^2}\right] \tag{2.18}$$

$$\beta=[\lambda+r]\left[\frac{\frac{\mu'}{2}(\lambda-1)^2[K_2'-(\lambda-1)^2]}{[K_2'-(\lambda-1)^2]^2+[K_1'(\lambda-1)]^2}+\frac{\frac{\mu}{2}\lambda^2[K_2-\lambda^2]}{[K_2-\lambda^2]^2+[K_1\lambda]^2}\right] \tag{2.19}$$

The matrix $\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}$ is simply the matrix $F_\infty$ of (B-31).

The characteristic equation for the system (2.17) correct to $0(\varepsilon)$ is

$$(\Lambda-\alpha)^2+\beta^2=0 \tag{2.20}$$

$$\Lambda^2-2\alpha\Lambda+\alpha^2+\beta^2=0 \tag{2.21}$$

The condition for stability is that

$$\alpha<0 \tag{2.22}$$

The stability condition may be written

$$\frac{(\lambda-1)^2\Delta'}{[K_2'-(\lambda-1)^2]^2+[K_1'(\lambda-1)]^2}+\frac{\lambda^2\Delta}{[K_2-\lambda^2]^2+[K_1\lambda]^2}>0 \tag{2.23}$$

where

$$\left.\begin{aligned}\Delta' &= \frac{\mu'}{2}K_1'(\lambda-1)(\lambda+r)\\ \Delta &= \frac{\mu}{2}K_1\lambda(\lambda+r)\end{aligned}\right\} \tag{2.24}$$

but

$$\left.\begin{aligned}&\lambda=\frac{(C-A)r+J}{A}\\ \therefore\quad &\lambda+r=\frac{Cr+J}{A}\\ \text{and}\quad &\lambda-1=\frac{(C-A)r+(J-A)}{A}\end{aligned}\right\} \tag{2.25}$$

Thus we have the following conditions:

i) System is asymptotically stable if

$\Delta,\Delta'>0$ (2.26)

ii) System unstable if

$\Delta,\Delta'<0$ (2.27)

iii) If $\Delta\,\Delta'\leq 0$, stability depends on the quantitative relationship between $\Delta$ and $\Delta'$

iv) In particular, if $\Omega$ the spin rate of the platform is zero, i.e. $r\equiv 0$

then $\Delta = \frac{\mu}{2} K_1 (\frac{J}{A})^2 > 0$ (2.28)

$$\Delta' = \frac{\mu'}{2} K_1' (\frac{J-A}{A})(\frac{J}{A}) \qquad (2.29)$$

Frequently $J<A$ $\therefore$ $\Delta'<0$ , however by making the dissipation in the platform sufficiently large, condition (2.22) can always be satisfied

v) Provided $\Delta>0$ , the dissipation in the platform may be maximized by setting $K_2=\lambda^2$ , in this case the condition for stability becomes:

$$\frac{\mu}{2} \frac{\lambda(\lambda+r)}{K_1} + \frac{\frac{\mu'}{2} K_1 (\lambda-1)^2 \Delta'}{[K_2'-(\lambda-1)^2]^2+[K_1'(\lambda-1)]^2} > 0 \qquad (2.30)$$

In Appendix B it is shown that the stability treatment presented above is rigorously correct for $\varepsilon=\mathrm{Max}(\mu,\mu')$ sufficiently small.

Other Problems

1) The technique above has also been used on the problem treated in reference (1) and the results agree exactly.

2) The technique above has also been applied to the case where the despin motor is used in conjunction with the products of inertia terms in the inertia tensor of the platform to obtain stability for the dual spin satellite.

# APPENDIX A

## Liapunov-Poincare Stability Theory

### Definitions

Given the dynamical system:

$$\left.\begin{aligned} &\frac{d\underline{x}}{dt}=A\underline{x}+\underline{f}(\underline{x},t) \\ &\underline{x}(0)=\underline{c} \\ &\lim_{||\underline{x}||\to 0} \frac{||\underline{f}(\underline{x},t)||}{||\underline{x}||}=0 \text{ uniformly in } t \end{aligned}\right\} \quad (A.1)$$

### Liapunov Stability

If given any $\delta>0$ there exists an $\varepsilon>0$ such that $||\underline{c}||\leq\varepsilon$ implies that $||\underline{x}(t)||\leq\delta$ , $\forall\, t>0$ , then the trivial solution of A.1 is said to be Liapunov Stable (L.S.)

### Liapunov Asymptotic Stability

If the trivial solution of A.1 is Liapunov stable and in addition $||\underline{x}(t)||$ tends to zero as $t$ tends to infinity, then the trivial solution of A.1 is said to be Liapunov Asymptotically Stable (L.A.S.)

### Liapunov Instability

If given a $\delta>0$ there exists no $\varepsilon>0$ such that $||\underline{c}||\leq\varepsilon$ implies that $||\underline{x}(t)||\leq\delta$ , $\forall\, t>0$ then the trivial solution of A.1 is said to be Unstable in the sense of Liapunov.

Theorem A1

If A is a stability matrix, i.e. if $\mathrm{Re}\lambda_i(A)<0\ \forall_i$, then A.1 is Liapunov asymptotically stable provided that $||\underline{c}||$ is sufficiently small.

Proof

Case (i) A nondefective

There exists a nonsingular matrix T such that $T^{-1}AT=\Lambda$ , $\mathrm{Re}\lambda_i<0\ \forall_i$

Let $\underline{x}=T\underline{z}$

Then

$$\left.\begin{aligned}
&\frac{d\underline{z}}{dt}=\Lambda\underline{z}+\underline{g}(\underline{z},t)\\
&\underline{z}(0)= =T^{-1}\underline{c}\\
&\underline{g}(\underline{z},t)=T^{-1}\underline{f}(T\underline{z},t)\\
&\lim_{||\underline{z}||\to 0}\frac{||\underline{g}(\underline{z},t)||}{||\underline{z}||}=0 \text{ uniformly in } t
\end{aligned}\right\} \tag{A.2}$$

Let

$$V(\underline{z})=\underline{z}^*\underline{z}=||\underline{z}||^2 \tag{A.3}$$

$$\dot{V}(\underline{z})=\dot{\underline{z}}^*\underline{z}+\underline{z}^*\dot{\underline{z}} \tag{A.4}$$

$$=\underline{z}^*(2\mathrm{Re}\Lambda)\underline{z}+2\mathrm{Re}\underline{z}^*\underline{g}(\underline{z},t) \tag{A.5}$$

$$=-\alpha V+\underline{z}^*(2\mathrm{Re}\Lambda+\alpha I)\underline{z}+2\mathrm{Re}\underline{z}^*\underline{g}(\underline{z},t) \tag{A.6}$$

If $0<\alpha<\underset{i}{\mathrm{Min}}\,(|\mathrm{Re}\lambda_i(A)|)$

Then

$$-[2\mathrm{Re}\Lambda+\alpha I]=Q=Q^T>0 \tag{A.7}$$

$$\therefore \qquad \dot{V}(z)=-\alpha V-W(\underline{z},t) \tag{A.8}$$

Where

$$W(\underline{z},t)=\underline{z}^*Q\underline{z}-2\mathrm{Re}\,\underline{z}^*\underline{g}(\underline{z},t) \tag{A.9}$$

Since $\underline{g}(\underline{z},t)$ contains no terms linear in $\underline{z}$, $W(\underline{z},t)>0$ provided $||\underline{z}||$ is sufficiently small.

Thus

$$\dot{V}(z)\leq-\alpha V \quad \text{for } ||\underline{z}|| \text{ sufficiently small} \tag{A.10}$$

Hence

$$V(z)\leq e^{-\alpha t}V(0) \tag{A.10}$$

Thus if $V(0)=||\underline{\mathscr{C}}||^2$ is sufficiently small, $V(z)=||\underline{z}||^2$ remains small and tends to zero as $t\to\infty$, hence (A.2) is Liapunov asymptotically stable. Since $\underline{x}=T\underline{z}$, stability of $\underline{z}$ implies stability of $\underline{x}$, hence (A.1) is L.A.S.

<u>Case (ii)</u> A-defective, in this case it is not possible to diagonalize A. However, there exists a nonsingular matrix T such that A can be reduced to Jordan Canonical form is

$$T^{-1}AT=\begin{bmatrix} J_{\alpha_1} & & \\ & J_{\alpha_2} & \\ & & J_{\alpha_k} \end{bmatrix}, \quad \sum_{i=1}^{k}\alpha_i=N \tag{A.11}$$

Where the $J_{\alpha_i}$ are Jordan blocks such as

$$J_{\alpha_i} = \begin{bmatrix} \lambda_i 1 & & & 0 \\ & \lambda_i 1 & & \\ & & \lambda_i 1 & \\ 0 & & & \lambda_i \end{bmatrix} \tag{A.12}$$

To simplify the proof, consider the case

$$J_{\alpha_1} = \begin{bmatrix} \lambda_1 & 1 \\ & \\ 0 & \lambda_1 \end{bmatrix} \qquad J_{\alpha_i} = \lambda_i \qquad i \geq 2$$

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ 0 & \lambda_1 & & \\ & & & \\ & 0 & & \Lambda_{N-2} \end{bmatrix} \tag{A.13}$$

Let $\underline{x} = T\underline{z}$

Then

$$\left.\begin{aligned} \frac{d\underline{z}}{dt} &= \begin{bmatrix} \lambda_1 & 1 & & 0 \\ 0 & \lambda_1 & & \\ & & & \\ & 0 & & \Lambda_{N-2} \end{bmatrix} \underline{z} + \underline{g}(\underline{z}, t) \\ \underline{z}(0) &= \underline{\mathscr{C}} = T^{-1}\underline{c} \end{aligned}\right\} \tag{A.14}$$

$$\left.\begin{aligned} &\underline{g}(\underline{z},t)=T^{-1}\underline{f}(T\underline{z},t) \\ &\lim_{||\underline{z}||\to 0} \frac{||\underline{g}(\underline{z},t)||}{||\underline{z}||}=0 \text{ uniformly in } t \end{aligned}\right\} \quad \text{(A.14) cont'd}$$

Let

$$V(\underline{z})=\underline{z}^*P\underline{z}>0 \tag{A.15}$$

Where

$$P=\begin{bmatrix} 1 & & 0 \\ & \dfrac{1}{(\mathrm{Re}\lambda_1)^2} & \\ 0 & & I \end{bmatrix}=P^T>0 \tag{A.16}$$

$$\dot{V}=\dot{\underline{z}}^*P\underline{z}+\underline{z}^*P\dot{\underline{z}} \tag{A.17}$$

$$=-\underline{z}^*Q\underline{z}+2\mathrm{Re}(\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.18}$$

Where

$$Q=\begin{bmatrix} -2\mathrm{Re}\lambda_1 & -1 & 0 \\ -1 & -\dfrac{2}{\mathrm{Re}\lambda_1} & \\ & 0 & -2\mathrm{Re}\Lambda_{N-2} \end{bmatrix} \tag{A.19}$$

Since $\mathrm{Re}\lambda_i<0 \quad \forall i$

$$Q=Q^T>0 \tag{A.20}$$

Thus

$$\dot{V}=-\alpha V-W(\underline{z},t) \tag{A.21}$$

$$\alpha>0$$

Where

$$W(\underline{z},t)=\underline{z}^*(Q-\alpha I)\underline{z}-2\mathrm{Re}\,\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.22}$$

If

$$\alpha < \frac{|\mathrm{Re}\lambda_1|}{(\mathrm{Re}\lambda_1)^2+1}\ ,\ \text{then } (Q-\alpha I) > 0 \tag{A.23}$$

Since $\underline{g}(\underline{z},t)$ contains no terms linear in $\underline{z}$, $W(\underline{z},t)$ is positive provided $||\underline{z}||$ is sufficiently small.

$$\therefore \qquad \dot{V} \le -\alpha V \quad \text{for } ||\underline{z}|| \text{ sufficiently small} \tag{A.24}$$

Applying the arguments of Case (i), we see that the trivial solution of (A.1) is Liapunov Asymptotically stable.

The technique developed above can easily be extended to cover the case of multiple repeated roots or higher order Jordan blocks.

Critical Cases

It will be observed that the techniques used to prove the stability of (A.1) breaks down if $\mathrm{Re}\lambda_i=0$ for $i \in (1,k)$, i.e. if the matrix A has one or more zero eigenvalue, or one or more pairs of complex conjugate pure imaginary eigenvalues. Such cases are called Critical Cases and will be treated in Theorem III.

Theorem AII

If the matrix A in (A.1) has one or more eigenvalue with positive real part, then the trivial solution of (A.1) is Liapunov unstable for sufficiently small initial data.

Where

$$W(\underline{z},t)=\underline{z}^*(Q-\alpha I)\underline{z}-2Re\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.22}$$

If

$$\alpha < \frac{|Re\lambda_1|}{(Re\lambda_1)^2+1}, \text{ then } (Q-\alpha I) > 0 \tag{A.23}$$

Since $\underline{g}(\underline{z},t)$ contains no terms linear in $\underline{z}$, $W(\underline{z},t)$ is positive provided $||\underline{z}||$ is sufficiently small.

$$\therefore \qquad \dot{V} \leq -\alpha V \quad \text{for} \quad ||\underline{z}|| \quad \text{sufficiently small} \tag{A.24}$$

Applying the arguments of Case (i), we see that the trivial solution of (A.1) is Liapunov Asymptotically stable.

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Theorem AII

If the matrix A in (A.1) has one or more eigenvalue with positive real part, then the trivial solution of (A.1) is Liapunov unstable for sufficiently small initial data.

Proof

Case (i) A non deflective, in this case there exists a nonsingular matrix $T$ which diagonalizes $A$.

i.e.

$$\left.\begin{aligned} &\text{Where} \quad T^{-1}AT=\Lambda \\ &Re\lambda_i > 0 \qquad i\in(1,k) \\ &Re\Lambda_j \le 0 \qquad j\in(k+1) \end{aligned}\right\} \tag{A.25}$$

Let $\underline{x}=T\underline{z}$

Then

$$\left.\begin{aligned} &\frac{dz}{dt}=\Lambda\underline{z}+\underline{g}(\underline{z},t) \\ &\underline{z}(0)= \ =T^{-1}\underline{c} \\ &\underline{g}(\underline{z},t)=T^{-1}\underline{f}(\underline{z},t) \\ &\lim_{||\underline{z}||\to 0} \frac{||\underline{g}(\underline{z},t)||}{||\underline{z}||}=0 \text{ uniformly in } t \end{aligned}\right\} \tag{A.26}$$

Let

$$V(\underline{z})=\underline{z}^*P\underline{z} \tag{A.27}$$

Where

$$P=\begin{bmatrix} I_k & 0 \\ & \\ 0 & -I_{N-k} \end{bmatrix}=P^T \tag{A.28}$$

$$\dot{V}=\underline{\dot{z}}^*P\underline{z}+\underline{z}^*P\underline{\dot{z}} \tag{A.29}$$

$$\dot{V}=\underline{z}^*Q\underline{z}+2\mathrm{Re}(\underline{z}^*P\underline{g}(\underline{z},t)) \tag{A.30}$$

Where

$$Q=\begin{bmatrix} 2\mathrm{Re}\Lambda k & 0 \\ & \\ 0 & -2\mathrm{Re}\Lambda_{N-k} \end{bmatrix}=Q^T\geq 0 \tag{A.31}$$

Hence

$$\dot{V}=\alpha V+W(\underline{z},t) \tag{A.32}$$

Where

$$W(z,t)=\underline{z}^*[Q-\alpha P]\underline{z}+2\mathrm{Re}\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.33}$$

$$\text{If} \quad 0<\alpha<\min_{1\leq i\leq k} \mathrm{Re}\,\lambda_i(A) \tag{A.34}$$

Then

$$(Q-\alpha P) \text{ is positive definite} \tag{A.35}$$

Since $\underline{g}(\underline{z},t)$ contains no term linear in $\underline{z}$, for $||\underline{z}||\leq\Delta$, sufficiently small, $W(\underline{z},t)$ is positive.

Hence

$$\dot{V}\geq\alpha V \tag{A.36}$$

$$\therefore \quad V(z)\geq e^{\alpha t}V(0) \tag{A.37}$$

Since $V(z)$ is sign indefinite, there exists a set $\Omega_1$:

$$\Omega_1 : V(\underline{z})\geq 0 \qquad \partial\Omega_1 : V(\underline{z})=0 \tag{A.38}$$

Define $\Omega_2$:

$$\Omega_2 : ||\underline{z}|| \le \delta < \Delta \tag{A.39}$$

Let

$$\Omega_3 : \Omega_1 \cap \Omega_2 \tag{A.39}$$

If

$$\underline{z}(0) \in \Omega_3 \;,\; V(0)>0 \tag{A.40}$$

$$\text{Then} \qquad V(z) \ge e^{\alpha t} V(0) > 0 \tag{A.41}$$

The trajectory $g^+$ cannot exit through $\partial\Omega_1$ since $V(0)>0$ and $V(z)$ is increasing, therefore it must exit through the boundary $||\underline{z}||=\delta$. Hence given any $0<\delta<\Delta$ there exists no $\varepsilon>0$, such that if $||\underline{z}(0)||\le\varepsilon$, $\underline{z}(0)\in\Omega_2$, $||\underline{z}(t)||\le\delta$ for $\forall\, t>0$ $\therefore$ the trivial solution of (A.1) is unstable in the sense of Liapunov.

<u>Case (ii)</u> A defective, in this case A cannot be diagonalized, however there exists a nonsingular matrix T which reduces A to Jordan Canonical form i.e.

$$T^{-1}AT = \begin{bmatrix} J_{\alpha 1} & & & \\ & J_{\alpha 2} & & \\ & & \ddots & \\ & & & J_{\alpha k} \end{bmatrix}, \quad \sum_{i=1}^{k} i = N \tag{A.42}$$

Where

$$J_{\alpha i}=\begin{bmatrix} \lambda_i 1 & & & \\ & \lambda_i 1 & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix} \qquad (A.43)$$

Let $\underline{x}=T\underline{z}$ in (A.1)

$$\left.\begin{aligned} &\frac{d\underline{z}}{dt}=J\underline{z}+\underline{g}(\underline{z},t) \\ &\underline{z}(0)=\underline{\mathscr{C}}=T^{-1}\underline{c} \\ &J=T^{-1}AT \\ &\underline{g}(\underline{z},t)=T^{-1}\underline{f}(T\underline{z},t) \\ &\lim_{||\underline{z}||\to 0} \frac{||\underline{g}(\underline{z},t)||}{||\underline{z}||}=0 \text{ uniformly in } t \end{aligned}\right\} \qquad (A.44)$$

To simplify the presentation we shall consider three typical cases

Case (iia)

$$J=\begin{bmatrix} \lambda_1 & 1 & & 0 \\ 0 & \lambda_1 & & \\ & & & \\ & 0 & & \Lambda_{N-2} \end{bmatrix} \qquad (A.45)$$

Where

$$\mathrm{Re}\lambda_1>0 \ , \ \mathrm{Re}\lambda_j\leq 0 \qquad j\in(3,N)$$

Let

$$V(\underline{z})=\underline{z}^*P\underline{z} \tag{A.46}$$

Where

$$P=\begin{bmatrix} 1 & & \\ & \dfrac{1}{(\mathrm{Re}\lambda_1)^2} & \\ & & -I_{N-k} \end{bmatrix} \tag{A.47}$$

Then

$$\dot{V}=\dot{\underline{z}}^*P\underline{z}+\underline{z}^*P\dot{\underline{z}} \tag{A.48}$$

$$=\underline{z}^*Q\underline{z}+2\mathrm{Re}\,\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.49}$$

Where

$$Q=Q^T=\begin{bmatrix} 2\mathrm{Re}\lambda_1 & 1 & \\ 1 & \dfrac{2}{\mathrm{Re}\lambda_1} & \\ & & -2\mathrm{Re}\Lambda_{N-k} \end{bmatrix} \geq 0 \tag{A.50}$$

equation (A.49) may be rewritten

$$\dot{V}=\alpha V+W(\underline{z},t) \; ; \; \alpha>0 \tag{A.51}$$

Where

$$W(\underline{z},t)=\underline{z}^*(Q-\alpha P)+2\mathrm{Re}\,\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.52}$$

If

$$0<\alpha<\frac{\mathrm{Re}\lambda_1}{1+(\mathrm{Re}\lambda_1)^2} \tag{A.53}$$

then

$$(Q-\alpha P) \text{ is positive definite} \tag{A.54}$$

Since $\underline{g}(\underline{z},t)$ contains no terms linear in $\underline{z}$, $W(\underline{z},t)$ is positive if $||\underline{z}|| \leq \Delta$, sufficiently small.

$$\therefore \qquad V \geq \alpha V \quad \text{for} \quad ||\underline{z}|| \leq \Delta \tag{A.55}$$

Since $V$ is sign indefinite, there exists a set $\Omega_1$,

$$\Omega_1 : V \geq 0 \ , \ V=0 \ \text{ on } \ \partial\Omega_1 \tag{A.56}$$

Define

$$\Omega_2 : \ ||\underline{z}|| \leq \delta < \Delta \qquad \text{(A.57}$$

$$\Omega_1 : \ \Omega_1 \cap \Omega_2 \tag{A.58}$$

From (A.55)

$$V(z) \geq e^{\alpha t} V(0) \tag{A.59}$$

If

$$\underline{z}(0) \in \Omega_3 \ , \exists V(0) > 0$$

Then $V(z)>0$ and monotone increasing provided $||\underline{z}|| \in \Omega_3$ .

The trajectory, $g^+$, starting in $\Omega_3$ with $V(0)>0$ cannot exit $\Omega_3$ through the boundary $\partial\Omega_1$, since $V=0$ on $\partial\Omega_1$, the trajectory must therefore exit through the boundary $||\underline{z}||=\delta$. Hence, given any $\delta$, $0<\delta<\Delta$, there exists no $\varepsilon>0$, such that $||\underline{z}(0)|| \leq \varepsilon$ implies $||\underline{z}(t)|| \leq \delta$ $\forall$ $t>0$. The trivial solution of (A.1) is therefore unstable in the sense of Liapunov.

Case (iib)

$$J=\begin{bmatrix} \Lambda_k & & & 0 \\ & \lambda_{k+1} & 1 & \\ & & \lambda_{k+1} & \\ 0 & & & \Lambda_{N-k-2} \end{bmatrix} \tag{A.60}$$

Where

$$\left.\begin{array}{ll} Re\lambda_i>0 & i\in(1,k) \\ Re(\lambda_{k+1})<0 & \\ Re\lambda_j\leq 0 & j\in(k+3,N) \end{array}\right\} \tag{A.61}$$

Let

$$V(\underline{z})=\underline{z}^*Pz \tag{A.62}$$

Where

$$P=\begin{bmatrix} I_k & & & \\ & -1 & & \\ & & \dfrac{-1}{(Re\lambda_{k+1})^2} & \\ & & & -I_{N-k-2} \end{bmatrix} \tag{A.63}$$

In this case

$$\dot{V}=\underline{z}^*Q\underline{z}+2Re\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.64}$$

Where

$$Q=\begin{bmatrix} 2\mathrm{Re}\lambda_k & & & 0 \\ & -2\mathrm{Re}\lambda_{k+1} & -1 & \\ & -1 & \dfrac{2}{-\mathrm{Re}\lambda_{k+1}} & \\ 0 & & & -2\mathrm{Re}\Lambda_{N-k-2} \end{bmatrix} \tag{A.65}$$

The matrix Q is clearly positive semi-definite.

Equation (A.64) may be rewritten

$$\dot{V}=\alpha V+W(\underline{z},t) \quad ; \quad \alpha>0 \tag{A.66}$$

Where

$$W(\underline{z},t)=\underline{z}^*(Q-\alpha P)\underline{z}+2\mathrm{Re}\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.67}$$

If

$$0<\alpha<\min_{1\le i\le k} \lambda_i \tag{A.68}$$

The $(Q-\alpha P)$ is positive definite and $W(\underline{z},t)$ is positive for $||\underline{z}|| \le \Delta$, sufficiently small. The arguments of Case (ii) apply here also and the trivial solution of (A.1) is unstable in the sense of Liapunov.

Case (iii)

$$J=\begin{bmatrix} \lambda_k & & 0 & \\ & \lambda_\ell & & \\ 0 & & 0 & 1 \\ & & 0 & 0 \end{bmatrix} \tag{A.69}$$

Where

$$\left.\begin{array}{ll} Re\lambda_i>0 & i\in(1,k) \\ Re\lambda_j\leq 0 & j\in(k+1,N-2) \end{array}\right\} \qquad (A.70)$$

Let

$$V(\underline{z})=\underline{z}^*P\underline{z} \qquad (A.71)$$

Where

$$P=\begin{bmatrix} I_k & & & 0 \\ & -I_\ell & & \\ & & -\beta & \\ 0 & & & -1 \end{bmatrix} \qquad (A.72)$$

Then

$$\dot{V}=\underline{z}^*Q\underline{z}+2Re\underline{z}^*\underline{g}(\underline{z},t) \qquad (A.73)$$

Where

$$Q=\begin{bmatrix} 2Re\Lambda_k & & & \\ & -2Re\Lambda_e & & \\ & & 0 & -1 \\ & & -1 & 0 \end{bmatrix} \qquad (A.74)$$

Equation (A.73) may be rewritten

$$\dot{V}=\alpha V+W(\underline{z},t) \quad ; \quad \alpha>0 \qquad (A.75)$$

Where

$$W(\underline{z},t)=\underline{z}^*(Q-\alpha P)\underline{z}+2\mathrm{Re}\,\underline{z}^*P\underline{g}(\underline{z},t) \tag{A.76}$$

If

$$\left.\begin{aligned} &0<\alpha<\min_{1\le i\le k} \lambda_i \\ &\beta=\frac{2}{\alpha^2} \end{aligned}\right\} \tag{A.77}$$

Then $(Q-\alpha P)$ is positive definite and $W(\underline{z},t)$ is positive for $||\underline{z}||\le\Delta$, sufficiently small. The arguments of Case (ii) apply here also and the trivial solution of (A.1) is unstable in the sense of Liapunov.

The techniques developed above are easily extended to the case of multiple repeated roots and higher order Jordan forms.

It should be noted in passing that unlike Theorem I, Theorem II does not break down in the case where one or more eigenvalues have a zero real part.

<u>Critical Cases in the Liapunov-Poincare Theory</u>

As already pointed out, if the matrix A has any eigenvalues with zero real part, stability cannot in general be inferred from the stability of the linearized equations. In the case of the attitude stability of satellites it will be shown that due to the special form of the equations of motion, stability of the full perturbational equations can still be inferred from the linearized or variational equations.

<u>Theorem AIII</u>

The perturbational equations governing the attitude stability of spinning satellites take the special form:

$$\left.\begin{aligned}
&\frac{d\underline{x}_1}{dt}=\underline{f}_1(\underline{x}_1,\underline{x}_2,t)\\
&\frac{d\underline{x}_2}{dt}=A\underline{x}_2+\underline{f}_2(\underline{x}_1,\underline{x}_2,t)\\
&\underline{x}=\begin{pmatrix}\underline{x}_1\\ \\ \underline{x}_2\end{pmatrix}\quad \underline{x}(0)=\underline{c}\\
&\underline{f}_1(\underline{x}_1,\underline{0},t)=\underline{0}\quad ;\quad \underline{f}_2(\underline{x}_1,0,t)=0\\
&\lim_{||\underline{x}_2||\to 0}\frac{||\underline{f}_1(\underline{x}_1,x_2,t)||}{||\underline{x}_2||}=0 \qquad \text{uniformly in } t
\end{aligned}\right\}\tag{A.78}$$

If the matrix $A$ is a stability matrix, then the trivial solution of (A.78) is Liapunov stable for sufficiently small initial data. Furthermore, the states $\underline{x}_1$ and $\underline{x}_2$ have the following properties

$$\left.\begin{aligned}
&\lim_{t\to\infty}||\underline{x}_2||\equiv 0\\
&\lim_{t\to\infty}||\underline{x}_1||=\gamma\ -\text{constant}
\end{aligned}\right\}\tag{A.79}$$

Proof

Let

$$\left.\begin{aligned}
&\underline{x}_1=\underline{z}_1\\
&\underline{x}_2=T\underline{z}_2\\
\text{Where}\qquad &T^{-1}AT=\Lambda\\
&\mathrm{Re}\lambda_i<0\quad \forall\ i
\end{aligned}\right\}\tag{A.80}$$

We shall only discuss the case where A is non-defective, the case for A defective is handled in a similar manner.

Using (A.80) equation (A.78) becomes:

$$\left.\begin{aligned}
&\frac{d\underline{z}_1}{dt}=\underline{g}_1(\underline{z}_1,\underline{z}_2,t)\\
&\frac{d\underline{z}_2}{dt}=\Lambda\underline{z}_2+\underline{g}_2(\underline{z}_1,z_2,t)\\
&\underline{z}(0)=\begin{pmatrix}\underline{z}_1(0)\\ \\ \underline{z}_2(0)\end{pmatrix}=\mathscr{C}=\begin{pmatrix}\underline{x}_1(0)\\ \\ T^{-1}\underline{x}_2(0)\end{pmatrix}\\
&\underline{g}_1(\underline{z}_1,\underline{z}_2,t)=\underline{f}_1(\underline{x}_1,T\underline{z}_2,t)\\
&\underline{g}_2(\underline{z}_1,\underline{z}_2,t)=T^{-1}\underline{f}_2(\underline{x}_1,T\underline{z}_2,t)\\
&\text{where}\quad\begin{cases}\underline{g}_1(\underline{z}_1,\underline{0},t)=0\\ \\ \underline{g}_2(\underline{z}_1,\underline{0},t)=\underline{0}\end{cases}\\
&\lim_{\|\underline{z}_2\|\to}\frac{\|\underline{g}_1(\underline{z}_1,\underline{z}_2,t)\|}{\|\underline{z}_2\|}=0\ \text{uniformly in}\ t
\end{aligned}\right\}\qquad\text{(A.81)}$$

Let

$$V(\underline{z})=V_1(\underline{z}_1)+V_2(\underline{z}_2)\qquad\text{(A.82)}$$

Where

$$\left.\begin{aligned} V_1(\underline{z}_1)&=\underline{z}_1^*\underline{z}_1=||\underline{z}_1||^2 \\ V_2(\underline{z}_2)&=\underline{z}_2^*\underline{z}_2=||\underline{z}_2||^2 \end{aligned}\right\} \quad (A.83)$$

Hence

$$V(\underline{z})=||\underline{z}_1||^2+||\underline{z}_2||^2=||\underline{z}||^2 \quad (A.84)$$

$$\dot{V}_2=\underline{z}_2^*2\mathrm{Re}\Lambda\underline{z}_2+2\mathrm{Re}\underline{z}_2^*\underline{g}_2(\underline{z}_1,\underline{z}_2,t) \quad (A.85)$$

(A.85) may be rewritten as

$$V_2=-\alpha V_2-W_2(\underline{z}_1,\underline{z}_2,t) \quad ; \quad \alpha>0 \quad (A.86)$$

Where

$$W_2(\underline{z}_1,\underline{z}_2,t)=\underline{z}_2^*(Q-\alpha I)\underline{z}_2-2\mathrm{Re}\underline{z}_2^*\underline{g}_2(\underline{z}_1,\underline{z}_2,t) \quad (A.87)$$

Where

$$Q=-2\mathrm{Re}\Lambda>0 \quad (A.88)$$

If

$$0<\alpha<\min_{1\leq i\leq N-2}|\mathrm{Re}\lambda_i| \quad (A.89)$$

Then

$$(Q-\alpha I) \text{ is positive definite} \quad (A.90)$$

Since $\underline{g}_2(\underline{z}_1,\underline{0},t)=\underline{0}$

$$\therefore \text{ if } \quad ||\underline{z}||=\sqrt{||\underline{z}_1||^2+||\underline{z}_2||^2}<\Delta \text{ sufficiently small} \quad (A.91)$$

Then

$$\left.\begin{aligned} &|2\mathrm{Re}\,\underline{z}_2^*\underline{g}_2(\underline{z}_1,\underline{z}_2,t)| \le K_2(\Delta)||\underline{z}_2||^2 \\ \text{Where} \quad &K_2(\Delta)\to 0 \quad \text{as} \quad \Delta\to 0 \end{aligned}\right\} \tag{A.92}$$

Thus by taking $\Delta$ sufficiently small

$$W_2(\underline{z}_1,\underline{z}_2,t) \ge \beta||\underline{z}_2||^2 > 0 \tag{A.93}$$

$$\therefore \quad \dot{V}_2 \le -\alpha V_2 \tag{A.94}$$

Hence

$$V_2(\underline{z}_2) \le e^{-\alpha t} V_2(0) \tag{A.95}$$

Hence if

$$||\underline{z}(0)|| \le \varepsilon < \Delta \tag{A.96}$$

$$\therefore \quad V_2(\underline{z}_2) \le e^{-\alpha t}\varepsilon^2 \tag{A.97}$$

$$\therefore \quad ||\underline{z}_2(t)|| \le e^{-\alpha/2t}\varepsilon \tag{A.98}$$

From (A.81)

$$\underline{z}_1 = \underline{z}_1(0) + \int_0^t \underline{g}_1(\underline{z}_1,\underline{z}_2,\tau)\,d\tau \tag{A.99}$$

Since $\underline{g}_1(\underline{z}_1,\underline{0},\tau)=0$ and $\underline{g}_1(\underline{z}_1,\underline{z}_2,\tau)$ satisfies a nonlinearity condition; uniformly in $t$

$$\left.\begin{aligned} &||\underline{g}_1(\underline{z}_1,\underline{z}_2,\tau)|| \le K_1(\Delta)V_2(\underline{z}_2) \\ &\text{for } ||\underline{z}|| \le \Delta \quad \text{and} \quad K_1(\Delta)\sim 0(1) \end{aligned}\right\} \tag{A.100}$$

Thus

$$||\underline{z}_1|| \leq ||\underline{z}_1,(0)||+K_1(\Delta) \int_0^t V_2(\underline{z}_2(t))dt \tag{A.101}$$

Using (A.97)

$$||\underline{z}_1(t)|| \leq ||\underline{z}_1(0)||+K_1(\Delta) \frac{1}{\alpha} (1-e^{-\alpha t})\varepsilon^2 \tag{A.102}$$

Using (A.96)

$$||\underline{z}_1(t)|| \leq K_2(\Delta)\varepsilon \tag{A.103}$$

$$\therefore \qquad V(\underline{z}(t))=||\underline{z}(t)||^2 \leq [K_2^2(\Delta)+e^{-\alpha t}]\varepsilon^2 \tag{A.104}$$

$$\therefore \qquad ||\underline{z}(t)|| \leq K_3(\Delta)\varepsilon \ , \quad \forall\, t>0 \tag{A.105}$$

Thus if we choose

$$\varepsilon \leq \varepsilon^* = \frac{\delta}{K_3(\Delta)} \tag{A.106}$$

Then given any $\delta$, $0<\delta<\Delta$ , if $||\underline{z}(0)|| \leq \varepsilon^*$ then $||\underline{z}(t)|| \leq \delta$ , $\forall\, t>0$ . Thus the trivial solution of the system (A.81) is Liapunov stable, and since (A.81) is derived from (A.78) by bounded linear transformations, system (A.78) is also Liapunov stable. In addition, using equation (A.98) we see that

$$\lim_{t\to\infty} ||\underline{z}_2(t)||=0 \tag{A.107}$$

Using (A.80), (A.107) implies that

$$\lim_{t\to\infty} ||\underline{x}_2(t)||=0 \tag{A.108}$$

From (A.99)

$$\lim_{t\to\infty} \underline{z}_1(t)=\underline{z}_1(0)+\int_0^{\infty} \underline{g}_1(z_1,z_2,\tau)d\tau \tag{A.109}$$

Since

$$\left|\left|\int_0^{\infty} \underline{g}_1(z_1,z_2,\tau)d\tau\right|\right| \leq \frac{K_1(\Delta)}{\alpha}\,\varepsilon^2 \quad ,$$

the integral (A.109) converges, hence, using (A.80)

$$\lim_{t\to\infty} ||\underline{x}_1(t)||=\gamma - \text{a constant} \tag{A.110}$$

provided that the initial data is sufficiently small.

<u>Extension of Theorem AIII to Systems with Periodic Coefficients</u>

In the study of the attitude stability of dual spin satellites, the perturbational equations take the following form:

$$\left.\begin{aligned}
&\frac{d\underline{x}_1}{dt} = \underline{f}_1(\underline{x}_1,\underline{x}_2,t)\\
&\frac{d\underline{x}_2}{dt} =A(t)\underline{x}_2+\underline{f}_2(\underline{x}_1,\underline{x}_2,t)\\
&\underline{x}=\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix} \qquad \underline{x}(0)=\underline{c}\\
&A(t+T)=A(t) \quad , \quad \forall\ t\\
&\underline{f}_i(\underline{x}_1,0,t)\equiv 0
\end{aligned}\right\} \tag{A.111}$$

$$\lim_{||\underline{x}_2||\to 0} \frac{||\underline{f}_1(\underline{x}_1,\underline{x}_2,t)||}{||\underline{x}_2||} = 0 \text{ uniformly in } t \qquad \text{(A.111) Cont'd}$$

For such systems, Theorem IV applies.

Theorem AIV

Given the system (A.111), if all the solutions of the equation

$$\frac{d\underline{x}_2}{dt} = A(t)\underline{x}_2 \qquad \underline{x}_2(0) = \underline{c}_2 \tag{A.112}$$

are Liapunov asymptotically stable, then the trivial solution of (A.111) is Liapunov stable for $||\underline{c}||$ sufficiently small. In addition the states $\underline{x}_1$ and $\underline{x}_2$ have the following properties.

$$\left.\begin{aligned} &\lim_{t\to\infty} ||\underline{x}_2|| = 0 \\ &\lim_{t\to\infty} ||\underline{x}_1|| = \gamma \text{ — a constant} \end{aligned}\right\} \tag{A.113}$$

Proof

Consider first the matrix equation

$$\frac{dX}{dt} = A(t)X \quad ; \quad X(0) = I \tag{A.114}$$

It is well known from Floquet theory that $X(t)$ has the following form:

$$X(t) = Q(t)e^{Bt} \tag{A.115}$$

Where $B$ is a constant matrix

and $Q(t+T) = Q(t)$ , $Q(0) = I$ is a bounded periodic matrix. (A.116)

The requirement that all solutions of equation (A.112) be Liapunov asymptotically stable is equivalent to the requirement that the matrix B be a stability matrix. i.e.

$$\mathrm{Re}\lambda_i(B)<0 \quad , \quad \forall\ i \tag{A.117}$$

The matrix $Q(t)$ in (A.115) satisfies the differential equation

$$\frac{dQ}{dt}+QB=A(t)Q \tag{A.118}$$

Consider now the Liapunov transformation

$$\underline{x}_2=Q(t)\underline{u}_2 \tag{A.119}$$

Substituting into (A.111)

$$\frac{dQ}{dt}\,\underline{u}_2+Q(t)\,\frac{d\underline{u}_2}{dt}=A(t)Q(t)\underline{u}_2+\underline{f}_2(\underline{x}_1,Q(t)\underline{u}_2,t) \tag{A.120}$$

$$\therefore \qquad \frac{d\underline{u}_2}{dt}=Q^{-1}(t)\left\{\left[A(t)Q(t)-\frac{dQ}{dt}\right]\underline{u}_2+\underline{f}_2(\underline{x}_1,Q(t)\underline{u}_2,t)\right\} \tag{A.121}$$

Using equation (A.118)

$$\left.\begin{array}{l} \dfrac{d\underline{u}_2}{dt}=B\underline{u}_2+\underline{h}_2(\underline{x}_1,\underline{u}_2,t) \\ \\ \text{where} \\ \\ \underline{h}_2(\underline{x}_1,\underline{u}_2,t)=Q^{-1}(t)\underline{f}_2(\underline{x}_1,Q(t)\underline{u}_2,t) \end{array}\right\} \tag{A.122}$$

Let

$$\underline{u}_1=\underline{x}_1 \tag{A.123}$$

Then system (A.111) becomes

$$\left.\begin{aligned}
&\frac{d\underline{u}_1}{dt}=\underline{h}_1(\underline{u}_1,\underline{u}_2,t)\\
&\frac{d\underline{u}_2}{dt}=B\underline{u}_2+\underline{h}_2(\underline{u}_1,\underline{u}_2,t)\\
&\underline{u}=\begin{pmatrix}\underline{u}_1\\ \\ \underline{u}_2\end{pmatrix}\qquad u(0)=\begin{pmatrix}\underline{x}_1(0)\\ \\ \underline{x}_2(0)\end{pmatrix}=\underline{c}\\
&\underline{h}_i(\underline{u}_1,0,t)\equiv 0\qquad i=1,2\\
&\lim_{||\underline{u}_2||\to 0}\frac{||\underline{h}_1(\underline{u}_1,\underline{u}_2,t)||}{||\underline{u}_2||}=0 \text{ uniformly in } t
\end{aligned}\right\}\qquad (A.124)$$

The system (A.124) has exactly the same structure as system (A.78), hence by Theorem III, the trivial solution of (A.124) is Liapunov stable for sufficiently small initial data and in addition $\underline{u}_1$ and $\underline{u}_2$ have the following properties:

$$\left.\begin{aligned}
&\text{i)}\quad \lim_{t\to\infty}||\underline{u}_2(t)||=0\\
&\text{ii)}\quad \lim_{t\to\infty}||\underline{u}_1(t)||=\gamma - \text{a constant}
\end{aligned}\right\}\qquad (A.125)$$

Using (A.119) and (A.123) it therefore follows that system (A.111) is Liapunov stable and $\underline{x}_1$ and $\underline{x}_2$ have the following properties:

$$\left.\begin{aligned}
&\text{a)}\quad \lim_{t\to\infty}||\underline{x}_2(t)||=0\\
&\text{b)}\quad \lim_{t\to\infty}||\underline{x}_1(t)||=\gamma - \text{a constant}
\end{aligned}\right\}\qquad (A.126)$$

Thus establishing Theorem IV.

## APPENDIX B

### Justification for the Method of Slowly Varying Parameters

The linearized equations of motion of dual-spin satellites with damping in both rotor and platform can be written in the following standard form:

$$\frac{d\underline{v}}{d\tau} = \lambda \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{v} + \varepsilon \left[ A_1(\tau) \frac{d\underline{x}}{d\tau} + A_2(\tau)\underline{x} \right] \tag{B.1}$$

$$\frac{d\underline{x}}{d\tau} = B_0\underline{x} + c(\tau) \frac{d\underline{v}}{d} + D(\tau)\underline{v} \tag{B.2}$$

Where $\underline{v}$ is a two vector, $\underline{x}$ is a four vector, $A_1(\tau)$, $A_2(\tau)$ , $B_0$ , $C(\tau)$, $D(\tau)$ are bounded matrices and $\varepsilon>0$ is a small parameter.

To reduce (A.1) and (A.2) to more convenient form, we introduce the following transformation

$$\underline{v} = \begin{bmatrix} \cos \lambda\tau & \sin \lambda\tau \\ \sin \lambda\tau & -\cos \lambda\tau \end{bmatrix} \underline{a} \tag{B.3}$$

Equation (A.1) becomes

$$\frac{d\underline{a}}{d\tau} = \varepsilon A_3(\tau) \frac{d\underline{x}}{d\tau} + \varepsilon A_4(\tau)\underline{x} \tag{B.4}$$

Where

$$\left.\begin{aligned} & A_3(\tau) = T(\lambda\tau)A_1(\tau) \\ & A_4(\tau) = T(\lambda\tau)A_2(\tau) \\ & T(\lambda\tau) = \begin{bmatrix} \cos \lambda\tau & \sin \lambda\tau \\ \sin \lambda\tau & -\cos \lambda\tau \end{bmatrix} \end{aligned}\right\} \tag{B.5}$$

Equation (B.2) becomes

$$\frac{d\underline{x}}{d\tau} = B_0\underline{x}\ C_1(\tau)\ \frac{d\underline{a}}{d\tau} + D_1(\tau)\underline{a} \tag{B.6}$$

where

$$\left.\begin{aligned} &C_1(\tau) = C(\tau)T(\lambda\tau) \\ &D_1(\tau) = C(\tau)\ \frac{dT}{d\tau} + D(\tau)T(\lambda\tau) \end{aligned}\right\} \tag{B.7}$$

Let us now introduce a second transformation

$$\left.\begin{aligned} &\underline{x} = \underline{z} + G_1(\tau)\underline{a} \\ &G_1(\tau) = \int_{-\infty}^{\tau} e^{B_0(\tau-\xi)} D_1(\xi)d\xi \end{aligned}\right\} \tag{B.8}$$

Substituting into (B.6)

$$\frac{d\underline{z}}{d\tau} = B_0\underline{z} + [C_1(\tau) - G_1(\tau)]\ \frac{d\underline{a}}{d\tau} \tag{B.9}$$

Substituting (B.6) into (B.4), using (B.8) and solving for $\frac{d\underline{a}}{d\tau}$

$$\frac{d\underline{a}}{d\tau} = \varepsilon F(\tau)\underline{a} + \varepsilon H_1(\tau)\underline{z} \tag{B.10}$$

Where

$$\left.\begin{aligned} &F(\tau) = [I - \varepsilon A_3 e_1]^{-1}[A_3 D_1 + (A_3 B_0 + A_4)G_1] \\ &H(\tau) = [I - \varepsilon A_3 C_1]^{-1}[A_3 B_0 + A_4] \end{aligned}\right\} \tag{B.11}$$

Substituting (B.10) into (B.9) we have

$$\frac{dz}{d\tau} + B_0\underline{z} + B_1(\tau)\underline{z} + \varepsilon H_2(\tau)\underline{a} \tag{B.12}$$

Where

$$\left.\begin{aligned} B_1(\tau) &= [C_1(\tau)-G_1(\tau)]H_1(\tau) \\ H_2(\tau) &= [C_1(\tau)-G_1(\tau)]F(\tau) \end{aligned}\right\} \tag{B.13}$$

For $\varepsilon$ sufficiently small, the matrices $F(\tau)$, $H_1(\tau)$, $H_2(\tau)$ and $B_1(\tau)$ are bounded.

If we write

$$F(\tau) = F_0 + F_1(\tau) \tag{B.14}$$

where

$$F_0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T F(\tau)d \tag{B.15}$$

<u>Theorem BI</u>

Given the system of equations

$$\left.\begin{aligned} \frac{da}{d\tau} &= \varepsilon[F_0+F_1(\tau)]\underline{a} + \varepsilon H_1(\tau)\underline{z} \\ \frac{dz}{d\tau} &= [B_0 + B_1(\tau)]\underline{z} + \varepsilon H_2(\tau)\underline{a} \end{aligned}\right\} \tag{B.16}$$

If i) $F_0$ and $B_0$ are stability matrices

ii) $\int F_1(\tau)d\tau$ is a bounded matrix

Then for $\varepsilon$ sufficiently small, the trivial solution of (B.16) is Liapunov asymptotically stable.

Proof

Since $F_0$ and $B_0$ are stability matrices there exist symmetric positive definite matrices $P_1$ and $P_2$ such that

$$\left.\begin{aligned} &F_0^T P_1 + P_1 F_0 = -2I_1 \\ &B_0^T P_2 + P_2 B_0 = -I_2 \end{aligned}\right\} \quad (B.17)$$

Consider the function

$$V = \underline{a}^T P_1 \underline{a} + \underline{z}^T P_2 \underline{z} - \varepsilon \underline{a}^T Q(\tau) \underline{a} \quad (B.18$$

Where

$$\left.\begin{aligned} &Q(\tau) = F_2^T(\tau) P_1 + P_1 F_2(\tau) \\ &F_2(\tau) = \int F_1(\tau) d\tau \end{aligned}\right\} \quad (B.19)$$

$Q(\tau)$ is a bounded matrix, since $P_1$ and $F_2(\tau)$ are bounded matrices

$$\dot{V} = \underline{\dot{a}}^T P_1 \underline{a} + \underline{a}^T P_1 \underline{\dot{a}} + \underline{\dot{z}}^T P_2 \underline{z} + \underline{z}^T P_2 \underline{\dot{z}} - \{\underline{a}^T \dot{Q} \underline{a} + \underline{\dot{a}}^T Q \underline{a} + \underline{a}^T Q \underline{\dot{a}}\} \varepsilon \quad (B.20.)$$

Using equation (B.16) and (B.19)

$$\left.\begin{aligned} \dot{V} = {} & \varepsilon \underline{a}^T [(F_0^T + F_1^T) P_1 + P_1 (F_0 + F_1)] \underline{a} + 2\varepsilon \underline{a}^T P_1 H_1 \underline{z} - \varepsilon [\underline{a}^T (F_1^T P_1 + P_1 F_1) \underline{a} + \varepsilon \underline{a}^T (F^T Q + QF) \underline{a} \\ & + 2\varepsilon \underline{a}^T Q H_1 \underline{z}] + \underline{z}^T [(B_0^T + \varepsilon B_1^T) P_2 + P_2 (B_0 + \varepsilon B_1)] \underline{z} + 2\varepsilon \underline{a}^T H_2^T P_2 \underline{z} \end{aligned}\right\} \quad (B.21)$$

Where

$$F+F_0+F_1 \tag{B.22}$$

Using (B.17) $\dot{V}$ becomes:

$$\dot{V}=-2\varepsilon\underline{a}^T\underline{a}-\underline{z}^T\underline{z}+\varepsilon\underline{z}^T(B_1^TP_2+P_2B_1)\underline{z}+2\varepsilon\underline{a}^T\underline{R}(\tau)\underline{z}-\varepsilon^2[\underline{a}^T(F^TQ+QF)\underline{a}+2\underline{a}^TQH_1\underline{z}] \tag{B.23}$$

Where

$$R(\tau)=[P_1H_1(\tau)+H_2^T(\tau)P_2] \tag{B.24}$$

Equation (B.23) may be rewritten:

$$\begin{aligned}\dot{V}=&-\varepsilon\underline{a}^T\underline{a}-\underline{z}^T[I-\varepsilon(B_1^TP_2+P_2B_1)-\varepsilon R_TR]\underline{z}\\ &-\varepsilon(\underline{a}-R\underline{z})^T(\underline{a}-R\underline{z})\\ &-\varepsilon^2[\underline{a}^T(F^TQ+QF)\underline{a}+2\underline{a}^TQH_1\underline{z}]\end{aligned} \tag{B.25}$$

Since $B_1,P_2,Q$, R etc. are bounded matrices, for $\varepsilon$ sufficiently small, the sign of $\dot{V}$ is that of the first three terms.

$$\therefore \qquad \dot{V}<0 \quad \text{for } \varepsilon \text{ sufficiently small} \tag{B.26}$$

Similarly, the sign of V, (B.18) is that of the first two terms for $\varepsilon$ sufficiently small. Hence, for $\varepsilon$ sufficiently small

$$V>0\ ,\ \dot{V}<0 \quad \therefore \quad V \text{ is a Liapunov function} \tag{B.27}$$

Hence, the trivial solution of equation (B.16) is Liapunov asymptotically stable.

Using (B.8), stability of $\underline{z}$ and $\underline{a}$ implies stability of $\underline{x}$ and $\underline{a}$ and hence of $\underline{x}$ and $\underline{v}$. Thus, under the hypothesis of Theorem (BI), the trivial solution of equations (B.1) and (B.2) is Liapunov asymptotically stable.

Given that $B_0$ is a stability matrix, the requirements for stability are that $\varepsilon$ be sufficiently small and that the time average of the matrix $F(\tau)$ should be a stability matrix.

Now

$$F(\tau)=[I-\varepsilon A_3(\tau)C_1(\tau)]^{-1}[A_3(\tau)B_0+A_4(\tau))G_1(\tau)+A_3(\tau)D_1(\tau)]$$

$$=(A_3(\tau)B_0+A_4(\tau))G_1(\tau)+A_3(\tau)D_1(\tau)$$

$$+\varepsilon[I-\varepsilon A_3C_1]^{-1}A_3C_1[(A_3B_0+A_4)G_1+A_3D_1 \tag{B.28}$$

Hence

$$F_0=F_{00}+\varepsilon F_{01}$$

Where

$$F_{00}=\lim_{T\to\infty}\frac{1}{T}\int_0^{\infty}\{(A_3B_0+A_4)G_1+A_3D\}d$$

$$F_{01}=\lim_{T\to\infty}\frac{1}{T}\int_0^{\infty}[I-\varepsilon A_3C_1]^{-1}A_3C_1[(A_3B_0+A_4)G_1+A_3D_1]d\tau \tag{B.29}$$

The requirement for stability is that $F_0$ be a stability matrix, for $\varepsilon$ sufficiently small this requirement will be satisfied if the matrix $F_{00}$ in (B.29) is a stability matrix.

In terms of the matrices $A_1$, $A_2$, $B_0$, $C$ and $D$, this condition becomes:

$$F_{00}=\lim_{T\to\infty} \frac{1}{T}\int_0^T T(\lambda\tau)[A_1B_0+A_2]\left\{\int_0^{\tau} e^{B_0(\tau-\xi)}\left(C\frac{d}{d\tau}+D\right)T(\lambda\xi)d\xi\right.$$

$$\left.+A_1[C\frac{d}{d\tau}+D]T(\lambda\tau)\right\} \tag{B.30}$$

should be a stability matrix.

Rather than carry out the operations in (B.30) in one step, $F_{00}$ may be evaluated in the following manner.

i) Make the transformation (B.3)

ii) Compute $\underline{x}_0(\tau)$ — the "steady state" response of equation (B.6) regarding $\underline{a}(\tau)$ as a constant vector

iii) Substitute the "steady state" response $\underline{x}_0(\tau)$ into equation (B.4)

iv) Time average equation (B.4) regarding $\underline{a}(\tau)$ as a "constant" vector

The procedure yields the equation

$$\frac{d\underline{a}}{d\tau}=F_{00}\underline{a} \tag{B.31}$$

Where $F_{00}$ is exactly the expression in equation (B.30).

It will be noted that this procedure is exactly what was done heuristically in Section 2 of this note.

# APPENDIX C

## Equations of Motion for a Dual-Spin Satellite

The dual-spin satellite consists of two rigid bodies having a common axis of rotation.

Let the axial moment of inertia of the rotor be J.

Let the total axial moment of inertia of the satellite be C (including platform, rotor and dampers)

Let the total equatorial moment of inertia of the satellite be A (including platform, rotor and dampers)

To simulate the effect of internal damping, the model will include torsional dampers.

Let $\psi$ be angle (about the common axis) between the body fixed axis in the rotor and platform.

### Equation of Motion

### Rotor

$$J[\dot{\omega}_3+\ddot{\psi}]-I_b'\dot{\theta}[\omega_2\cos\psi-\omega_1\sin\psi]=T_M-T_B \tag{C.1}$$

Where $I_b'$ is the polar moment of inertia of the damper on the rotor

$\theta$ is the rotation angle of the torsional damper on the rotor

$T_M$ the torque of the despin motor

$T_B$ the frictional torque of the rotor bearings.

Rotor/Platform

$$\left.\begin{aligned}
&C\dot{\omega}_3+J\ddot{\psi}-I_b'\dot{\theta}(\omega_2\cos\psi-\omega_1\sin\psi)-I_b'\dot{\phi}\omega_2=0\\
&A\dot{\omega}_1+[(C-A)\omega_3+J\dot{\psi}]\omega_2+I_b'[\ddot{\theta}\cos\psi-\dot{\theta}(\omega_3+\dot{\psi})\sin\psi]+I_b\ddot{\phi}=0\\
&A\dot{\omega}_2-[(C-A)\omega_3+J\dot{\psi}]\omega_1+I_b'[\ddot{\theta}\sin\psi+\dot{\theta}(\omega_3+\dot{\psi})\cos\psi]+I_b\omega_3\dot{\phi}=0
\end{aligned}\right\}\qquad (C.2)$$

Where $I_b$ is the polar moment of inertia of the torsional damper on the platform $\phi$ is the rotation angle of that damper.

Dampers

$$\left.\begin{aligned}
&I_b'\ddot{\theta}+\bar{K}_1'\dot{\theta}+\bar{K}_2'\theta=-I_b'[\dot{\omega}_1\cos\psi+\dot{\omega}_2\sin\psi+\omega_2\dot{\psi}\cos\psi-\omega_1\dot{\psi}\sin\psi]\\
&I_b\ddot{\theta}+\bar{K}_1\dot{\phi}+\bar{K}_2 \;=-I_b\dot{\omega}_1
\end{aligned}\right\}\qquad (C.3)$$

Steady State Solution $(T_M=T_B)$

$$\left.\begin{aligned}
&\omega_3=\Omega\;,\;\dot{\psi}=0\;,\;\psi=\sigma t\\
&\theta=\phi=\dot{\theta}=\dot{\phi}=\omega_1=\omega_2\equiv 0
\end{aligned}\right\}\qquad (C.4)$$

Perturbed Motion

Let

$$\left.\begin{aligned}
&\omega_3=\Omega+\sigma\xi\;;\;\tau=\sigma t\;;\;r=\frac{\Omega}{\sigma}\\
&\psi=\qquad \psi_2=\dot{\psi}_1=\sigma(1+\zeta)\\
&\frac{\omega_1}{\sigma}=v_1\;;\;\frac{\omega_2}{\sigma}=v_2\qquad T_B-T_M=\beta\,\sigma\zeta
\end{aligned}\right\}\qquad (C.5)$$

Perturbational Equations

$$\left.\begin{aligned}
&\frac{J}{A}\left[\frac{d\xi}{d\tau}+\frac{d\zeta}{d\tau}\right]-\mu'\frac{d\theta}{d\tau}\left[v_2\cos(\tau+\eta)-v_1\sin(\tau+\eta)\right]+\frac{B}{A}\zeta=0\\
&\frac{d\eta}{d\tau}=\zeta\\
&\frac{d\xi}{d\tau}+\frac{J}{C}\frac{d\zeta}{d\tau}-\mu'\frac{A}{C}\left[v_2\cos(\tau+\eta)-v_1\sin(\tau+\eta)\right]\frac{d\theta}{d\tau}-\mu\frac{A}{C}v_2\frac{d\phi}{d\tau}=0\\
&\frac{dv_1}{d\tau}+\left[\lambda+\frac{(C-A)\xi+J\zeta}{A}\right]v_2+\mu\frac{d^2\theta}{d\tau^2}+\mu'\left[\frac{d^2\theta}{d\tau^2}\cos(\tau+\eta)-\frac{d\theta}{d\tau}(1+r+\xi+\zeta)\sin(\tau+\eta)\right]=0\\
&\frac{dv_2}{d\tau}-\left[\lambda+\frac{(C-A)\xi+J\zeta}{A}\right]v_1+\mu(r+\xi)\frac{d\theta}{d\tau}+\mu'\left[\frac{d^2\theta}{d\tau^2}\sin(\tau+\eta)+\frac{d\theta}{d\tau}(1+r+\xi+\zeta)\cos(\tau+\eta)\right]=0\\
&\frac{d^2\theta}{d\tau^2}+K_1'\frac{d\theta}{d\tau}+K_2'\theta+\left[\left(\frac{dv_1}{d\tau}+v_2(1+\zeta)\right)\cos(\tau+\eta)+\left(\frac{dv_2}{d\tau}-v_1(1+\zeta)\right)\sin(\tau+\eta)\right]=0\\
&\frac{d^2\phi}{d\tau^2}+K_1\frac{d\phi}{d\tau}+K_2\phi+\frac{dv_1}{d\tau}=0
\end{aligned}\right\}\quad(C.6)$$

Where

$$\left.\begin{aligned}
&\mu=\frac{I_b}{A}\;;\;\mu'=\frac{I_b'}{A}\\
&K_1=\frac{K_1}{I_b\sigma}\;;\;K_1'=\frac{\overline{K}_1'}{I_b'\sigma}\\
&K_2=\frac{\overline{K}_2}{I_b\sigma^2}\;;\;K_2'=\frac{\overline{K}_2}{I_b'\sigma^2}
\end{aligned}\right\}\quad(C.7)$$

Let

$$\underline{\xi} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{C.8}$$

$$\underline{p} = \begin{pmatrix} v_1 \\ v_2 \\ \phi \\ \frac{d\phi}{d\tau} \\ \theta \\ \frac{d\theta}{d\tau} \end{pmatrix} \tag{C.9}$$

Using (C.8) and (C.9), equations (C.6) can be rewritten as:

$$\frac{d\underline{\xi}}{dt} = A_1\underline{\xi} + \underline{f}_1(\underline{\xi},\underline{p},\tau) \tag{C.10}$$

$$\frac{d\underline{p}}{dt} = A_2\underline{p} + \mu[B_1(\tau) + B_2(\underline{\xi},\underline{p},\tau)]\,\frac{d\underline{p}}{dt} + \underline{f}_2(\underline{\xi},\underline{p},\tau) \tag{C.11}$$

Where

$$\left.\begin{array}{l}
A_1,\ A_2 \quad \text{are constant matrices} \\[2ex]
B_1(\tau+2\pi) = B_1(\tau) \\[2ex]
\displaystyle\lim_{\|\underline{\xi}\|+\|\underline{p}\|\to 0} \frac{\|B_2(\underline{\xi},\underline{p},\tau)\|}{\|\underline{\xi}\|+\|\underline{p}\|} = 0 \ \text{uniformly in } \ t \\[3ex]
\underline{f}_1(\underline{\xi},\underline{0},\tau) = \underline{0} \\[2ex]
\underline{f}_2(\underline{\xi},0,\tau) = \underline{0}
\end{array}\right\} \tag{C.12}$$

$$\left.\begin{aligned}
&\lim_{||\underline{p}||\to 0} \frac{||\underline{f}_1(\underline{\xi},\underline{p},\tau)||}{||\underline{p}||}=0 \\
&\lim_{||\underline{\xi}||+||\underline{p}||\to 0} \frac{||\underline{f}_2(\underline{\xi},\underline{p},\tau)||}{||\underline{\xi}||+||\underline{p}||}=0
\end{aligned}\right\} \quad \begin{matrix}\text{(C.12)}\\ \text{cont'd}\end{matrix}$$

Solving (C.11) for $d\underline{p}/d\tau$, we have:

$$\frac{d\underline{p}}{d\tau}=A_3(\tau)\underline{p}+\underline{f}_3(\underline{\xi},\underline{p},\tau) \tag{C.13}$$

Where

$$\left.\begin{aligned}
A_3(\tau+\pi)&=A_3(\tau) \\
&=[I-\mu B_1(\tau)]^{-1}A_2 \\
\underline{f}_3(\underline{\xi},\underline{p}\tau) &=[I-\mu B_1(\tau)]^{-1}\underline{f}_2(\underline{\xi},p,\tau) \\
&+\mu[I-\mu(B_1(\tau)+B_2(\underline{\xi},\underline{p},\tau)]^{-1}B_2(\underline{\xi},\underline{p},\tau)A_3\underline{p}
\end{aligned}\right\} \quad \text{(C.14)}$$

Where

$$\left.\begin{aligned}
&\underline{f}_3(\underline{\xi},0,\tau)=\underline{0} \\
\text{and} \quad &\lim_{||\underline{\xi}||+||\underline{p}||\to 0} \frac{||\underline{f}_3(\underline{\xi},\underline{p},\tau)||}{||\underline{\xi}||+||\underline{p}||}=0 \text{ uniformly in } t
\end{aligned}\right\} \quad \text{(C.15)}$$

Let

$$\underline{\xi}=T\underline{y} \quad \text{where} \quad T^{-1}AT=\begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & -\gamma \end{bmatrix} \tag{C.16}$$

$$\gamma>0$$

Equations (C.10) and (C.13) become

$$\left.\begin{aligned}
\frac{d\underline{y}}{dt} &= \begin{bmatrix} 0 & & \\ & 0 & \\ & & -\gamma \end{bmatrix} \underline{y}+\underline{f}_4(\underline{y},\underline{p},\tau) \\[2em]
\frac{d\underline{p}}{dt} &= A_3(\tau)\underline{p}+\underline{f}_5(\underline{y},\underline{p},\tau)
\end{aligned}\right\} \quad (C.17)$$

Let

$$\underline{x}_1=\begin{pmatrix} y_1 \\ \\ y_2 \end{pmatrix} \;;\; \underline{x}_2=\begin{pmatrix} y_3 \\ \\ \underline{p} \end{pmatrix} \;,\; \underline{x}=\begin{pmatrix} \underline{x}_1 \\ \\ \underline{x}_2 \end{pmatrix} \qquad (C.18)$$

Equations (C.17) can now be written in the form:

$$\left.\begin{aligned}
&\frac{d\underline{x}_1}{d\tau} = \underline{\overline{f}}_1(\underline{x}_1,\underline{x}_2,\tau) \\[1em]
&\frac{d\underline{x}_2}{d\tau} = A(\tau)\underline{x}_2+\underline{\overline{f}}_2(\underline{x}_1,\underline{x}_2,\tau) \\[1em]
&\underline{x}=\begin{pmatrix} \underline{x}_1 \\ \\ \underline{x}_2 \end{pmatrix} \qquad \underline{x}(0)=\underline{c} \\[1em]
&A(\tau+\pi)=A(\tau) \\[1em]
&\underline{\overline{f}}_i(\underline{x}_1,0,\tau)=0 \qquad\qquad i = 1, 2 \\[1em]
&\lim_{||\underline{x}_2||\to 0} \frac{||\underline{\overline{f}}_1(\underline{x}_1,\underline{x}_2,\tau)||}{||\underline{x}_2||} = 0 \text{ uniformly in } \tau
\end{aligned}\right\} \quad (C.19)$$

Where

$$A(\tau)=\begin{bmatrix} -\gamma & \\ & \\ & A_3(\tau) \end{bmatrix} \tag{C.20}$$

Theorem IV of Appendix A is applicable to (C.19) and stability is guaranteed, for $||\underline{c}||$ sufficiently small, provided all solutions of

$$\frac{d\underline{x}_2}{d\tau}=A(\tau)\underline{x}_2 \tag{C.21}$$

are Liapunov asymptotically stable.

Using (C.20), all solutions of (C.21) will be asymptotically provided all solutions of

$$\frac{d\underline{p}}{d\tau}=A_3(\tau)\underline{p} \tag{C.22}$$

are asymptotically stable.

Since equation (C.22) is the linear part of equation (C.13), the condition for stability can be expressed in terms of the variational equations obtained by linearizing the perturbational equations (C.6).

Thus, the conditions for stability of a dual-spin satellite are:

1) The initial perturbations shall be sufficiently small
2) All solutions of the following set of differential equations shall be asymptotically stable

$$\left.\begin{aligned}
&\frac{dv_1}{d\tau}+\lambda v_2+\mu\frac{d^2\phi}{d\tau^2}+\mu'\left[\frac{d^2\theta}{d\tau^2}\cos\tau-(1+r)\frac{d\theta}{d\tau}\sin\tau\right]=0\\
&\frac{dv_2}{d\tau}-\lambda v_1+\mu r\frac{d\phi}{d\tau}=\mu'\left[\frac{d^2\theta}{d\tau^2}\sin\tau+(1+r)\frac{d\theta}{d\tau}\cos\tau\right]=0\\
&\frac{d^2\theta}{d\tau^2}+K_1'\frac{d\theta}{d\tau}+K_2'\theta+\left[\left(\frac{dv_1}{d\tau}+v_2\right)\cos\tau+\left(\frac{dv_2}{d\tau}-v_1\right)\sin\tau\right]=0\\
&\frac{d^2\phi}{d\tau^2}+K_1\frac{d\phi}{d\tau}+K_2\phi+\frac{dv_1}{d\tau}=0
\end{aligned}\right\}\quad (C.23)$$

It will be noted that the present analysis rigorously justifies the normal engineering analysis, in which one examines only the stability of the linearized equations and ignores completely the subtleties of the stability of the perturbational equations.

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